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On Spectral Tangential Nevanlinna–Pick Interpolation*

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In this paper we extend our work on spectral matricial Nevanlinna–Pick interpolation from [4] to the tangential case. As in [4] this will be deduced as a corollary of a spectral commutant lifting theorem which generalizes the corresponding result of [6].

1. INTRODUCTION

In our previous paper [4], motivated by a number of problems in control engineering, we proved a spectral generalization of the commutant lifting theorem which allowed us to extend classical matricial Nevanlinna–Pick interpolation (in which one bounds the norm of the interpolants) to certain spectral interpolation problems. The purpose of the present note is to extend this work still further to include the spectral analogue of the tangential interpolation results as considered in Fedcina [6].

In order to describe our results, let us briefly consider classical Nevanlinna–Pick theory in the matrix case. For $z_j \in D$ distinct (D denotes the unit disc), $1 \leqslant j \leqslant n$, let $F_1, ..., F_n$ be $N \times N$ matrices. Then we are

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interested in finding necessary and sufficient conditions for the existence of an analytic (in the disc D) $N \times N$ matrix-valued function $F(z)$ with $F(z_j) = F_j$ $(1 \leqslant j \leqslant n)$, and such that $\|F\|_\infty \leqslant 1$. It is well known [1, 2, 8] that the existence of F can be reduced to the determination of the positivity of a certain Hermitian "Nevanlinna–Pick" matrix. (This fact can be deduced, e.g., from the commutant lifting theorem [9, 10, 11].) In the paper [4], we studied the problem of bounding the *spectral radius* of the interpolating functions. In fact, we gave necessary and sufficient conditions for the existence of an interpolating F whose spectral radius is bounded away from 1. This was derived as a consequence of a spectral commutant lifting result.

Now in many control problem (see, e.g., [5, 7, 13]) one is interested in a variant of the above problem which was first studied classically by Fedcina [6]. This problem may be formulated as follows: Let $u_j, v_j \in \mathbf{C}^N$ be non-zero vectors. Then we want necessary and sufficient conditions for the existence of an analytic (in D) $N \times N$ matrix-valued function F such that $F(z_j)\, u_j = v_j$ for $1 \leqslant j \leqslant n$ and such that $\|F\| \leqslant 1$. This is the problem of *tangential Nevanlinna–Pick* interpolation. Fedcina [6] shows that again this question reduces to determining the positivity of a certain Hermitian matrix. In this note, we will solve the analogous spectral interpolation problem, where we do not necessarily require that the norm of the interpolating function F be bounded by 1, but instead its spectral radius. As in [4], this will be deduced as a consequence of a general spectral commutant lifting theorem.

We should note that the tangential spectral problem is in a certain sense easier to solve than the full matricial case considered in [4], and this paper provides a rather complete description of the optimal solutions which could certainly be implemented on computer. This of course is very important for some of the control engineering applications which originally motivated this work.

We now summarize the contents of this note. In Section 2, we set up some basic notation and quote several results from [4, 12]. In Section 3, we formulate and prove our tangential spectral commutant lifting theorem which is applied in Section 4 to prove the spectral tangential Nevanlinna–Pick theorem. Finally, we give in Section 5 an explicit algorithm for finding the optimal interpolants.

2. Preliminary Remarks

Throughout this paper, by "Hilbert space" we will mean "complex separable Hilbert space." By "operator" we shall always mean "bounded linear operator," unless explicitly stated otherwise. For $\mathscr{H}$ a Hilbert space, let $B(\mathscr{H})$ denote the set of operators on $\mathscr{H}$. For $\mathscr{H}_1$ and $\mathscr{H}_2$ Hilbert spaces, we set

$$B(\mathscr{H}_1, \mathscr{H}_2) := \{A : \mathscr{H}_1 \to \mathscr{H}_2 : A \text{ an operator}\}.$$

Let $\mathscr{E}$ be a Hilbert space, and denote by

$$H^2(\mathscr{E}) := H^2 \otimes \mathscr{E}$$

the Hilbert space of square-summable $\mathscr{E}$-valued power series (see [3, 11]). Given a bounded analytic function $F: D \to B(\mathscr{E})$, we can consider the multiplication operator $M_F: H^2(\mathscr{E}) \to H^2(\mathscr{E})$ defined by

$$(M_F f)(z) := F(z) f(z), \qquad f \in H^2(\mathscr{E}), \qquad z \in D.$$

The operator M_F commutes with the unilateral shift S given by

$$(Sf)(z) := zf(z), \qquad f \in H^2(\mathscr{E}),\ z \in D,$$

and it satisfies the norm equality

$$\|M_F\| = \|F\|_\infty = \sup\{\|F(z)\| : z \in D\}.$$

By slight abuse of notation we shall sometimes identify F and M_F in what follows when no confusion will be possible.

Given an operator A, $\|A\|_{\text{sp}}$ will denote its spectral radius. We will now state without proof two results from [4] which we will be using implicitly throughout the paper:

PROPOSITION 1. *Let $F: \bar{D} \to B(\mathscr{E})$ be a continuous function, analytic in D. Then*

$$\|M_F\|_{sp} = \sup_{z \in D} \|F(z)\|_{sp} = \max_{z \in \bar{D}} \|F(z)\|_{sp}.$$

PROPOSITION 2. *If $\mathscr{E}$ is finite dimensional, and $F: D \to B(\mathscr{E})$ is a bounded analytic function, then*

$$\|M_F\|_{sp} = \sup\{\|F(z)\|_{sp} : z \in D\}.$$

For $L: \mathscr{H} \to \mathscr{H}$ a contraction ($\mathscr{H}$ is a Hilbert space), let $D_L := (I - L^*L)^{1/2}$, and $\mathscr{D}_L := \overline{D_L \mathscr{H}}$. Then we will be using the following proposition in Section 5 whose proof may be easily derived from the results of [12]:

PROPOSITION 3. *Let $A : \mathscr{H}_1 \oplus \mathscr{H}_2 \to \mathscr{H}_1 \oplus \mathscr{H}_2$ be an operator (where $\mathscr{H}_1$, $\mathscr{H}_2$ are Hilbert spaces). Suppose that $A\mathscr{H}_1 \subset \mathscr{H}_1$, so that we can express*

$$A = \begin{bmatrix} B & X \\ 0 & Y \end{bmatrix},$$

for $B := A \mid \mathscr{H}_1$, $Y :\times (A^* \mid \mathscr{H}_2)^*$, *and* $X: \mathscr{H}_2 \to \mathscr{H}_1$. *Then we have that*:

(i) A *is a contradiction if and only if* $\|B\| \leqslant 1$, $\|Y\| \leqslant 1$, *and* $X = D_{B^*} C D_Y$ *for some operator* $C: \mathscr{D}_Y \to \mathscr{D}_{B^*}$, $\|C\| \leqslant 1$.

(ii) *If* $\|B\| < 1$, $\|Y\| < 1$, $X = D_{B^*} C D_Y$ *with* $C: \mathscr{D}_Y \to \mathscr{D}_{B^*}$ *and* $\|C\| < 1$, *then* $\|A\| < 1$.

Remark 1. In [4] we have shown that it is not possible to significantly weaken the hypotheses Propositions 1 and 2. Indeed, one can show that if the analytic function $F: D \to B(\mathscr{E})$ fails to be continuous on $\bar{D}$, and if $\mathscr{E}$ is infinite dimensional, then we may have

$$\sup_{z \in D} \|F(z)\|_{\mathrm{sp}} < \|M_F\|_{\mathrm{sp}}.$$

3. Generalized Spectral Commutant Lifting Theorem

In this section, we will define the main object of study of this paper , and prove our main result. We use the notation and terminology of Section 2 here.

In order to motivate our results, let us first briefly review the set-up from [4]. Accordingly, let $\mathscr{H}$ be a Hilbert space, $T \in \mathscr{B}(\mathscr{H})$, and let $A \in \{T\}'$ denotes the commutant of T. In [4], we defined the T-spectral radius $\rho_T(A)$ of A as

$$\rho_T(A) := \inf\{\|X^{-1}AX\| : X \text{ invertible, } X \in \{T\}'\}.$$

The T-spectral radius was used in the formulation of the spectral commutant lifting theorem from [4].

For the study of the tangential Nevanlinna–Pick problem, we will need a generalization of $\rho_T(A)$. For $T \in B(\mathscr{H})$, let $\mathscr{M} \subset \mathscr{H}$ be a subspace which is T^*-invariant. We denote by $P_{\mathscr{M}}$ the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$, and by $T_{\mathscr{M}}$ the compression of T to $\mathscr{M}$, i.e.,

$$T_{\mathscr{M}} := P_{\mathscr{M}} T \mid \mathscr{M} = (T^* \mid \mathscr{M})^*.$$

Let now $A \in \mathscr{L}(\mathscr{H})$ be an operator such that $A\mathscr{H} \subset \mathscr{M}$ and $T_{\mathscr{M}} A = AT$ or equivalently

$$P_{\mathscr{M}} TA = AT. \tag{1}$$

We want to introduce a quantity analogous to $\rho_T(A)$. To do this note that if $X \in \{T\}'$ is invertible, then $\mathscr{M}_1 := X^* \mathscr{M}$ is an invariant subspace for T^*.

In fact

$$\mathscr{M}_1 = X^* \mathscr{M} = \mathscr{H} \ominus X^{-1}(\mathscr{H} \ominus \mathscr{M}), \tag{2}$$

and it is easily seen that

$$P_{\mathscr{M}_1} X^{-1} = P_{\mathscr{M}_1} X^{-1} P_{\mathscr{M}}, \; P_{\mathscr{M}} X = P_{\mathscr{M}} X P_{\mathscr{M}_1}. \tag{3}$$

We claim that the operator $A_1 := P_{\mathscr{M}_1} X^{-1} A X$ satisfies

$$P_{\mathscr{M}_1} T A_1 = A_1 T.$$

Indeed, using $P_{\mathscr{M}_1} T P_{\mathscr{M}_1} = P_{\mathscr{M}_1} T$, (1) and (3), we obtain

$$\begin{aligned} A_1 T &= P_{\mathscr{M}_1} X^{-1} A X T \\ &= P_{\mathscr{M}_1} X^{-1} A T X \\ &= P_{\mathscr{M}_1} X^{-1} P_{\mathscr{M}} T A X \\ &= P_{\mathscr{M}_1} X^{-1} T A X \\ &= P_{\mathscr{M}_1} T X^{-1} A X \\ &= P_{\mathscr{M}_1} T P_{\mathscr{M}_1} X^{-1} A X \\ &= P_{\mathscr{M}_1} T A_1 \end{aligned}$$

as desired.

Moreover, A can be recovered from A_1 by the formula

$$A = P_{\mathscr{M}} X A_1 X^{-1}.$$

Indeed, using the second relation in (3), we see that

$$\begin{aligned} P_{\mathscr{M}} X A_1 X^{-1} &= P_{\mathscr{M}} X P_{\mathscr{M}_1} X^{-1} A X X^{-1} \\ &= P_{\mathscr{M}} X P_{\mathscr{M}_1} X^{-1} A \\ &= P_{\mathscr{M}} X X^{-1} A \\ &= P_{\mathscr{M}} A \\ &= A. \end{aligned}$$

Motivated by the above observations, it seems natural to define

$$\rho_{T,\mathscr{M}}(A) := \inf\{\|P_{X^*\mathscr{M}} X^{-1} A X\| : X \text{ is invertible and } X \in \{T\}'\}.$$

Note that for $\mathscr{H} = \mathscr{M}$, we have

$$\rho_{T,\mathscr{M}}(A) = \rho_T(A).$$

From this point on, it will be assumed that $\mathscr{H}$ is contained in a larger Hilbert $\mathscr{K}$ equipped with an isometry U such that

$$U^*\mathscr{H} \subset \mathscr{H}$$

and

$$T = U_{\mathscr{H}} := P_{\mathscr{H}} U | \mathscr{H}.$$

Given an operator $A: \mathscr{H} \to \mathscr{M}$ such that

$$P_{\mathscr{M}} TA = AT,$$

the commutant lifting theorem [10, 11] shows that there exist operators $B \in \{U\}'$ with

$$P_{\mathscr{M}} B = AP_{\mathscr{H}}.$$

For such A, we set

$$\mathrm{Dil}(A) := \{B \in \{U\}' : P_{\mathscr{M}} B = AP_{\mathscr{H}}\}.$$

Now let

$$\tau(A) := \inf\{\|B\|_{\mathrm{sp}} : B \in \mathrm{Dil}(A)\},$$

where $\|B\|_{\mathrm{sp}}$ denotes as above the spectral radius of B.

The key to our solution of the tangential spectral Nevanlinna–Pick problem is the following result:

THEOREM 1 (Tangential Spectral Commutant Lifting Theorem). *Let $U \in B(\mathscr{K})$ be an isometry, $\mathscr{H} \subset \mathscr{K}$ a finite dimensional hyperinvariant subspace for U^*, $T = U_{\mathscr{H}}$, and $\mathscr{M} \subset \mathscr{H}$ an invariant subspace for T^*. Then for every operator $A \in B(\mathscr{H})$ such that $A\mathscr{H} \subset \mathscr{M}$ and $P_{\mathscr{M}} TA = AT$, we have*

$$\tau(A) = \rho_{T,\mathscr{M}}(A).$$

Proof. The proof is similar to that given in [4] for the spectral commutant lifting theorem, but requires a few careful modifications. Given $\varepsilon > 0$, fix $B \in \mathrm{Dil}(A)$ such that $\|B\|_{\mathrm{sp}} < \tau(A) + \varepsilon$. By [4, Proposition 1], there exists an invertible $Y \in \{U\}'$ such that

$$\|Y^{-1}BY\| < \tau(A) + \varepsilon.$$

Since $\mathscr{H}$ is hyperinvariant for U^*, the operator $X = Y_{\mathscr{H}} := P_{\mathscr{H}} Y | \mathscr{H}$ commutes with T, is invertible, and $X^{-1} = (Y^{-1})_{\mathscr{H}}$. With the notation $\mathscr{M}_1 = X^*\mathscr{M}$, using the fact that $B \in \mathrm{Dil}(A)$ and relation (3), we have

$$\begin{aligned}
\|P_{\mathcal{M}_1}X^{-1}AX\| &= \|P_{\mathcal{M}_1}X^{-1}AP_{\mathcal{H}}X\| \\
&= \|P_{\mathcal{M}_1}X^{-1}(P_{\mathcal{M}}B\,|\,\mathcal{H})X\| \\
&= \|P_{\mathcal{M}_1}X^{-1}(P_{\mathcal{H}}B\,|\,\mathcal{H})X\| \\
&= \|P_{\mathcal{M}_1}(Y^{-1})_{\mathcal{H}}B_{\mathcal{H}}Y_{\mathcal{H}}\| \\
&\leqslant \|(Y^{-1}BY)_{\mathcal{H}}\| \\
&\leqslant \|Y^{-1}BY\| \\
&< \tau(A)+\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we see that

$$\rho_{T,\mathcal{M}}(A) \leqslant \tau(A).$$

For the converse, again let $\varepsilon > 0$ and choose an invertible $X \in \{T\}'$ such that

$$\|P_{\mathcal{M}_1}X^{-1}AX\| < \rho_{T,\mathcal{M}}(A)+\varepsilon.$$

Once more from [4] (see Lemma 1), there exists an invertible $Y \in \{U\}'$ such that $X = Y_{\mathcal{H}}$. As we noted above, if we set

$$A_1 = P_{\mathcal{M}_1}X^{-1}AX,$$

we have $T_{\mathcal{M}_1}A_1 = A_1T$ and the commutant lifting theorem implies the existence of $B_1 \in \mathrm{Dil}(A_1)$ satisfying

$$\|B_1\| = \|A_1\| < \rho_{T,\mathcal{M}}(A)+\varepsilon. \tag{4}$$

Set

$$B := YB_1Y^{-1}$$

and note that clearly

$$\|B\|_{\mathrm{sp}} \leqslant \|B_1\| < \rho_{T,\mathcal{M}}(A)+\varepsilon.$$

We claim that $B \in \mathrm{Dil}(A)$. Indeed, we know that

$$A = P_{\mathcal{M}}XA_1X^{-1},$$

so that

$$\begin{aligned}
AP_{\mathcal{H}} &= P_{\mathcal{M}}XA_1X^{-1}P_{\mathcal{H}} \\
&= P_{\mathcal{M}}XA_1P_{\mathcal{H}}X^{-1}P_{\mathcal{H}} \\
&= P_{\mathcal{M}}XP_{\mathcal{M}_1}B_1P_{\mathcal{H}}X^{-1}P_{\mathcal{H}}
\end{aligned}$$

$$
\begin{aligned}
&= P_{\mathcal{M}} X P_{\mathcal{H}} B_1 P_{\mathcal{H}} X^{-1} P_{\mathcal{H}} \\
&= P_{\mathcal{M}} Y_{\mathcal{H}} (B_1)_{\mathcal{H}} P_{\mathcal{H}} (Y^{-1})_{\mathcal{H}} P_{\mathcal{H}} \\
&= P_{\mathcal{M}} (Y B_1 Y^{-1})_{\mathcal{H}} P_{\mathcal{H}} \\
&= P_{\mathcal{M}} B P_{\mathcal{H}} \\
&= P_{\mathcal{M}} B,
\end{aligned}
$$

where we have used the second equality in (3), and the fact that $B(\mathcal{K} \ominus \mathcal{H}) \subset \mathcal{K} \ominus \mathcal{H}$. We conclude that

$$\tau(A) \leqslant \|B\|_{\mathrm{sp}} < \rho_{T, \mathcal{M}}(A) + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary,

$$\tau(A) \leqslant \rho_{T, \mathcal{M}}(A),$$

and the theorem is proved. ∎

Remark 2. We would like to make some comments now on the applicability of Theorem 1. Note that for T completely non-unitary (i.e., T has no non-zero reducing subspaces on which it is unitary; see [3, 11]), U must be a shift of finite multiplicity. In this case, up to unitary equivalence, we may assume that U is the canonical shift on $H^2(\mathcal{E}) = H^2 \otimes \mathcal{E}$, where $\mathcal{E}$ is a finite dimensional (complex) Hilbert space. (All of our Hardy spaces H^j, $1 \leqslant j \leqslant \infty$, will be defined on D in the standard way.) The hyper-invariant subspaces of $H^2(\mathcal{E})$ with respect to U^* have the form $H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$ with $m \in H^\infty$ inner (see [3, 11]), and the space $H^2(\mathcal{E}) \ominus mH^2(\mathcal{E})$ is finite dimensional if and only if m is a finite Blaschke product. Thus the operators T to which Theorem 1 applies have the form $S(m) \otimes I_{\mathcal{E}}$, with $\mathcal{E}$ a finite dimensional Hilbert space. (Recall that for $P_{H(m)}\colon H^2 \to H^2 \ominus mH^2$ orthogonal projection, and for S the canonical shift on H^2, $S(m) := P_{H(m)} S \mid H(m)$.)

4. Tangential Spectral Nevanlinna–Pick Theory

In this section, we apply Theorem 1 to a spectral version of the tangential Nevanlinna–Pick interpolation problem. In order to do this, we first put the Nevanlinna–Pick theory into the commutant lifting framework [9–11, 3]. Accordingly, we recall the general problem of tangential Nevanlinna–Pick interpolation.

Let $z_1, ..., z_n \in D$ be distinct, let $u_1, u_2, ..., u_n \in \mathbf{C}^N$ be non-zero vectors, and let $v_1 . v_2, ..., v_n \in \mathbf{C}^N$ be arbitrary vectors. We are interested in those bounded analytic functions $F: D \to B(\mathbf{C}^N)$ which satisfy the interpolation conditions

$$F(z_j)\, u_j = v_j \tag{5}$$

for $j = 1, ..., n$. The classical tangential Nevanlinna–Pick problem asks for such functions F with

$$\|F\|_\infty := \sup\{\|F(z)\| : z \in D\} \leqslant 1,$$

while the spectral problems asks for such a function F satisfying

$$\|M_F\|_{\text{sp}} = \sup\{\|F(z)\|_{\text{sp}} : z \in D\} \leqslant 1.$$

(See Proposition 2 above.)

We will construct an isometry $U \in \mathscr{L}(\mathscr{K})$, a finite dimensional hyper-invariant subspace $\mathscr{H}$ for U^*, an invariant subspace $\mathscr{M}$ for $T^* := U^* \mid \mathscr{H}$, and an operator $A: \mathscr{H} \to \mathscr{M}$ such that $T_{\mathscr{M}} A = AT$ and with the following property: there exists an isometric bijection between $\operatorname{Dil}(A)$ and the interpolating functions satisfying (5).

The first observation is that replacing F by $G(z) := F(\bar{z})^*$ condition (5) becomes

$$G(\bar{z}_j)^* u_j = v_j. \tag{6}$$

Let

$$\mathscr{K} := H^2 \otimes \mathbf{C}^N,$$

$$m(z) := \prod_{j=1}^{n} \frac{z - \bar{z}_j}{1 - z_j z},$$

and set $\mathscr{H} := H(m) \otimes \mathbf{C}^N$. Further, let U denote the canonical shift on $\mathscr{K}$, and T the compression of U to $\mathscr{H}$. To define the space $\mathscr{M}$, we need to consider the vectors $y_j \in H(m)$ given by

$$y_j(z) := \frac{1}{1 - z_j z}, \qquad z \in D.$$

We have

$$S(m)^* y_j = z_j\, y_j, \qquad j = 1, ..., n,$$

where as above S is the canonical shift on H^2, and $S(m)$ is the compression of S to $H(m)$. Note that $U = S \otimes I_{\mathbf{C}^N}$, and $T = S(m) \otimes I_{\mathbf{C}^N}$.

Define now

$$\mathscr{M} := \mathbf{C} y_1 \otimes u_1 + \cdots + \mathbf{C} y_n \otimes u_n.$$

Finally, we define an operator

$$A(z_1, ..., z_n; u_1, v_1; ...; u_n, v_n): \mathscr{H} \to \mathscr{M}$$

by setting

$$A^*(z_1, ..., z_n; u_1, v_1; ...; u_n, v_n)(y_j \otimes u_j) := y_j \otimes v_j, \qquad 1 \leqslant j \leqslant n. \tag{7}$$

We will sometimes set $A(u_1, v_1; ...; u_n, v_n) := A(z_1, ..., z_n; u_1, v_1; ...; u_n, v_n)$ when the dependence on the z_j is clear, or even $A := A(z_1, ..., z_n; u_1, v_1; ...; u_n, v_n)$ when the dependence on the z_j, u_j, v_j is understood.

Since A^* takes eigenvectors of $T^* | \mathscr{M}$ to eigenvectors corresponding to the same eigenvalue of T^*, the relation

$$T^*A^* = A^*T^* | \mathscr{M}$$

is immediate. Thus we have that

$$P_{\mathscr{M}} TA = AT.$$

It is well known that the commutant $\{U\}'$ consists of analytic multiplication (Toeplitz) operators of the form M_G, where $G: D \to B(\mathbf{C}^N)$ is bounded and analytic. We can now state:

LEMMA 1. *With the above notation, let $G: D \to B(\mathbf{C}^N)$ be analytic and bounded. Then the interpolation conditions* (6) *are satisfied if and only if $M_G \in \mathrm{Dil}(A)$.*

Proof. It suffices to verify that

$$M_G^*(y_j \otimes u) = y_j \otimes G(\bar{z}_j)^* u$$

for every $j = 1, ..., n$ and $u \in \mathbf{C}^n$. To do this. first note that for $u \in \mathbf{C}^N$, $h \in H^2(\mathbf{C}^N)$ we have

$$\langle h, y_j \otimes u \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{\langle h(e^{i\theta}), u \rangle}{1 - e^{-i\theta} \bar{z}_j} \, d\theta = \langle h(\bar{z}_j), u \rangle.$$

Now

$$\begin{aligned} \langle h, M_G^*(y_j \otimes u) \rangle &= \langle M_G h, y_j \otimes u \rangle \\ &= \langle G(\bar{z}_j)\, h(\bar{z}_j), u \rangle \\ &= \langle h(\bar{z}_j), G(\bar{z}_j)^* u \rangle \\ &= \langle h, y_j \otimes G(\bar{z}_j)^* u \rangle \end{aligned}$$

as claimed. ∎

Next note that if $X \in \{T\}'$, then necessarily

$$X^*(y_j \otimes \eta) = y_j \otimes X_j \eta$$

for all $\eta \in \mathbf{C}^N$ and where $X_j \in B(\mathbf{C}^N)$ for $j = 1, ..., n$. X is invertible if and only if each X_j $(1 \leqslant j \leqslant n)$ is invertible. Moreover, we have that

$$XA^*(u_1, v_1; ...; u_n, v_n) X^{-1} | X\mathcal{M} = A^*(X_1 u_1, X_1 v_1; ..., X_n u_n, X_n v_n).$$

Clearly

$$\rho_{T,\mathcal{M}}(A) = \inf\{\|A(X_1 u_1, X_1 v_1; ..., X_n u_n, X_n v_n)\| : X_j \in B(\mathbf{C}^N), X_j \text{ invertible}\}. \tag{8}$$

Now define the *tangential Nevanlinna–Pick matrix* as

$$\mathcal{N}(z_1, ..., z_n; u_1, v_1; ...; u_n, v_n; \rho) := \left[\frac{\langle \rho u_j, \rho u_k \rangle - \langle v_j, v_j \rangle}{1 - \bar{z}_j z_k}\right]_{1 \geqslant j,k \leqslant n}$$

We can now state the following spectral analogue of the main result of [6]:

THEOREM 2 (Tangential Nevanlinna–Pick Theorem). *There exists $F \in H^\infty(\mathbf{C}^N)$ with $\|M_F\|_{\text{sp}} < \rho$ satisfying the interpolation conditions* (5) *if and only if there exist $X_j \in B(\mathbf{C}^N)$ $(1 \leqslant j \leqslant n)$ invertible such that*

$$\mathcal{N}(z_1, ..., z_n; X_1 u_1, X_1 v_1; ...; X_n u_n, X_n v_n; \rho) > 0.$$

Proof. We have that

$$\|A(X_1 u_1, X_1 v_1; ...; X_n u_n v_n)\| < \rho$$

if and only if

$$\rho^2 I - A(X_1 u_1, X_1 v_1; ...; X_n u_n, X_n v_n) A^*(X_1 u_1, X_1 v_1; ...; X_n u_n, X_n v_n) > 0,$$

i.e., if and only if

$$\mathcal{N}(z_1, ..., z_n; X_1 u_1, X_1 v_1; ...; X_n u_n, X_n v_n; \rho) > 0.$$

The required conclusion now follows from Theorem 1. ∎

We now would like to discuss the dependence of $\rho_{T,\mathcal{M}}(A)$ on the given interpolation data. Specifically, set for $1 \leqslant k \leqslant n$

$$\sigma_k := \inf\{\|A(X_1 u_1, X_1 v_1; ...; X_k u_k, X_k v_k)\| : X_j \in B(\mathbf{C}^N), X_j \text{ invertible}, 1 \leqslant j \leqslant k\}. \tag{9}$$

We can now state:

PROPOSITION 4. *Suppose that u_n and v_n are linearly independent. Then*

$$\sigma_{n-1} = \sigma_n.$$

Proof. One could use the tangential Nevanlinna–Pick theorem to prove this result, but we prefer the following rather straightforward argument. Clearly $\sigma_{n-1} \leqslant \sigma_n$. Suppose that $\sigma_{n-1} < \sigma_n$. Now from Theorem 1, there exists $F \in H^\infty(\mathbf{C}^{N \times N})$ such that

$$F(z_j)\, u_j = v_j, \qquad 1 \leqslant j \leqslant n-1$$

with $\|F\|_{\mathrm{sp}} < \sigma_n$. Suppose that $F(z_n) \notin \mathbf{C} I_{\mathbf{C}^N}$. In this case, there exist linearly independent vectors u and v such that

$$F(z_n) u = v.$$

But since u_n and v_n are linearly independent, we can always find an invertible matrix X_n such that $X_n u_n = u$ and $X_n v_n = v$, which implies by our above discussion that

$$\sigma_n \leqslant \|M_F\|_{\mathrm{sp}} < \sigma_n,$$

a contradiction.

To complete the proof then we must show that we can always arrange interpolating F with $\|M_F\|_{\mathrm{sp}} < \sigma_n$ to be such that $F(z_n)$ is not a constant multiple of the identity. But this is easy. Suppose that $F(z_n)$ were such a constant multiple. Then we can find an analytic (in the unit disc) rational $N \times N$ matrix-valued function R of arbitrarily small norm such that $R(z_n)$ is not a constant multiple of the identity, and which vanishes at $z_1, z_2, ..., z_{n-1}$. Replacing F by $F + R$ we see that we have completed the proof of the proposition. ▮

Remark 3. Proposition 4 means that in spectral tangential Nevanlinna–Pick interpolation, we can ignore points z_j such that u_j and v_j are linearly independent vectors. Therefore from now on without loss of generality we will assume that

$$v_j = \lambda_j u_j, \qquad \lambda_j \neq 0, \quad j = 1, ..., n.$$

We now have the following result:

PROPOSITION 5. *Suppose that $n \leqslant N$. Then*

$$\sigma_n = \max\{|\lambda_1|, ..., |\lambda_n|\}.$$

Proof. Clearly

$$\sigma_n \geqslant \max\{|\lambda_1|, ..., |\lambda_n|\}.$$

We can obtain the required equality by choosing the X_j such that the $X_j u_j$ are orthogonal. ∎

Remarks 4. (i) Thus from Propositions 4 and 5, we see that as long as the number of dependent pairs (u_j, v_j) is $\leqslant N$, we have a rather easy way of computing σ_n. In the next section, we consider the case in which the number of dependent vector pairs exceeds N.

(ii) We would like to give now a simple example which shows that the infimum might not be a minimum in the definition of $\rho_{T,\mathcal{M}}$ when some of the pairs u_j, v_j are linearly independent. Specifically, let $z_1 = 0$, $z_2 = \frac{1}{2}$, let u_1 be any non-zero vector, $v_1 = 0$, and let u_2 and v_2 be linearly independent. Clearly, in this case $\rho_{T,\mathcal{M}} = 0$. On the other hand, the zero interpolating function is obviously not a solution. We will see in Section 5 that when all the pairs are linearly dependent, then in fact it is possible to replace the "inf" with "min" in (8).

5. Optimal Solutions

In this section we consider the spectral tangential Nevanlinna–Pick interpolation problem in which u_j and v_j are linearly independent for each j. As we saw before the general case actually reduces to this one. Fix therefore $z_1, ..., z_n \in D$ and $\lambda_1, ..., \lambda_n \in \mathbf{C}$ such that $v_j = \lambda_j u_j$, $\lambda_j \neq 0$ for $1 \leqslant j \leqslant n$. We set

$$A_{u_1, ..., u_n} := A(z_1, ..., z_n; u_1\, v_1; ...; u_n, v_n)$$

in this case. As in Section 4, we have that

$$\rho_{T,\mathcal{M}} := \rho_{T,\mathcal{M}}(A_{u_1,...,u_n}) = \inf\{\|A_{X_1 u_1, ..., X_n u_n}\| : X_j \in B(\mathbf{C}^N),\ X_j \text{ invertible}\}.$$

In the latter infimum only the vectors $w_j = X_j u_j$ count, so that

$$\begin{aligned}\rho_{T,\mathcal{M}}(A_{u_1,...,u_n}) &= \inf\{\|A_{w_1,...,w_n}\| : w_1, ..., w_n \in \mathbf{C}^N \setminus \{0\}\} \\ &= \inf\{\|A_{w_1,...,w_n}\| : w_j \in \mathbf{C}^N,\ \|w_j\| = 1,\ 1 \leqslant j \leqslant n\}.\end{aligned}$$

Since the unit sphere in $\mathbf{C}^N$ is compact, the latter infimum is actually attained. The operator $A_{u_1,...,u_n}$ will be said to be *optimal* if

$$\rho_{T,\mathcal{M}}(A_{u_1,...,u_n}) = \|A_{u_1,...,u_n}\|.$$

Our discussion shows that for the case at hand (in which all the pairs u_j,

v_j are linearly dependent) optimal operators do indeed exist. See also Remark 4(ii).

Fix for the moment $u_1, ..., u_n$, and as above let $\mathscr{M}$ be the space generated by

$$\{y_i \otimes u_i : 1 \leqslant i \leqslant n\}.$$

We have that

$$A^*_{u_1, ..., u_n} = \lambda_i\, y_i \otimes u_i$$

so that $A^*_{u_1, ..., u_n}$ can be considered for all practical purposes to be an operator acting on $\mathscr{M}$.

Our idea now is to fix j, $1 \leqslant j \leqslant n$, and vary the vector u_j while we keep $\{u_i : i \neq j\}$ fixed. In order to do this, we let $\mathscr{M}_j$ denote the space generated by $\{y_i \otimes u_i : i \neq j\}$, and note that the restriction

$$B_j := A^*_{u_1, ..., u_{j-1}, u, u_{j+1}, ..., u_n} \mid \mathscr{M}_j$$

does no depend on u for each $1 \leqslant j \leqslant n$. Therefore, if we write

$$\mathscr{M}_u = \mathscr{M}_j + \mathbf{C} y_j \otimes u = \mathscr{M}_j \oplus \mathscr{L}_u$$

for each $u \in \mathbf{C}^N$, $u \neq 0$, we see that $A^*_{u_1, ..., u_{j-1}, u, u_{j+1}, ..., u_n}$ has a decomposition of the form

$$A^*_{u_1, ..., u_{j-1}, u, u_{j+1}, ..., u_n} = \begin{bmatrix} B_j & D_u \\ 0 & C_u \end{bmatrix}.$$

Let $P_{\mathscr{M}_j} : H^2 \otimes \mathbf{C}^N \to \mathscr{M}_j$ denote orthogonal projection. Then to determine C_u and D_u, we note that $\mathscr{L}_u$ is generated by the single vector $(I - P_{\mathscr{M}_j})(y_j \otimes u)$, and

$$\begin{aligned} &A^*_{u_1, ..., u_{j-1}, u, u_{j+1}, ..., u_n}(I - P_{\mathscr{M}_j})(y_j \otimes u) \\ &\quad = \lambda_j (y_j \otimes u) - B_j P_{\mathscr{M}_j}(y_j \otimes u) \\ &\quad = \lambda_j (I - P_{\mathscr{M}_j})(y_j \otimes u) + (\lambda_j - B_j)\, P_{\mathscr{M}_j}(y_j \otimes u). \end{aligned}$$

We conclude that C_u is simply multiplication by λ_j, while D_u sends the unit vector

$$(I - P_{\mathscr{M}_j})(y_j \otimes u) / \|(I - P_{\mathscr{M}_j})(y_j \otimes u)\|$$

to

$$(\lambda_j - B_j)\, P_{\mathscr{M}_j}(y_j \otimes u) / \|(I - P_{\mathscr{M}_j})(y_j \otimes u)\|.$$

Identifying $\mathcal{M}_j \oplus \mathcal{L}_u$ with $\mathcal{M}_j \oplus \mathbf{C}$, we may write

$$A^*_{u_1, \ldots, u_{j-1}, u, u_{j+1}, \ldots, u_n} = \begin{bmatrix} B_j & \dfrac{(\lambda_j - B_j)\, P_{\mathcal{M}_j}(y_j \otimes u)}{\|(I - P_{\mathcal{M}_j})(y_j \otimes u)\|} \\ 0 & \lambda_j \end{bmatrix}.$$

LEMMA 2. *Let $\rho > 0$. We have $\|A^*_{u_1, \ldots, u_{j-1}, u, u_{j+1}, \ldots, u_n}\| \leqslant \rho$ if and only if the following conditions are satisfied*:

(i) $\|B_j\| \leqslant \rho$;

(ii) $|\lambda_j| \leqslant \rho$;

(iii) *there exists a vector $f_{\rho, u} \in \mathcal{M}_j$ such that $\|f_{\rho, u}\| \leqslant 1$ and*

$$(\lambda_j - B_j)\, P_{\mathcal{M}_j}(y_j \otimes u)/\rho\, \|(I - P_{\mathcal{M}_j})(y_j \otimes u)\| \\ = (1 - |\lambda_j|^2/\rho^2)^{1/2}(1 - B_j B_j^*/\rho^2)^{1/2} f_{\rho, u}.$$

Proof. This follows immmediately from Proposition 3 above. ∎

Remark 5. Note that in case $\rho > \max(\|B_j\|, |\lambda_j|)$, then

$$f_{\rho, u} = \frac{\rho(\rho^2 - B_j B_j^*)^{-1/2}(\lambda_j - B_j)\, P_{\mathcal{M}_j}(y_j \otimes u)}{(\rho^2 - |\lambda_j|^2)^{1/2} \|(I - P_{\mathcal{M}_j})(y_j \otimes u)\|}.$$

Since $y_j \otimes u \notin \mathcal{M}_j$, we have $(I - P_{\mathcal{M}_j})(y_j \otimes u) \neq 0$ if $u \neq 0$.

We now have:

LEMMA 3. *Let $u_j \in \mathbf{C}^N \setminus \{0\}$ be such that*

$$\|A_{u_1, u_2, \ldots, u_n}\| = \inf\{\|A_{u_1, \ldots, u_{j-1}, u, u_{j+1}, \ldots, u_n}\| : u \in \mathbf{C}^N \setminus \{0\}\} =: \rho_j.$$

Suppose that

$$\rho_j > \max\{\|B_j\|, |\lambda_j|\}.$$

Then

$$\rho_j^2 P_{y_j \otimes \mathbf{C}^N}(\lambda_j - B_j)^*(\rho_j^2 - B_j B_j^*)^{-1}(\lambda_j - B_j)\, P_{\mathcal{M}_j}(y_j \otimes u_j) \\ = (\rho_j^2 - |\lambda_j|^2)\, P_{y_j \otimes \mathbf{C}^N}(I - P_{\mathcal{M}_j})(y_j \otimes u_j).$$

Proof. Let $X, Y : y_j \otimes \mathbf{C}^N \to H^2 \otimes \mathbf{C}^N$ be given by

$$X(y_j \otimes u) := \rho_j(\rho_j^2 - B_j B_j^*)^{-1/2}(\lambda_j - B_j)\, P_{\mathcal{M}_j}(y_j \otimes u)$$

and

$$Y(y_j \otimes u) := (\rho_j^2 - |\lambda_j|^2)^{1/2}\, (I - P_{\mathcal{M}_j})(y_j \otimes u)$$

for $u \in \mathbf{C}^N$. Note that

$$\|f_{\rho_j,u}\| = \frac{\|X(y_j \otimes u)\|}{\|Y(y_j \otimes u)\|}.$$

By Lemma 2 and Proposition 3, we have that

$$\frac{\|X(y_j \otimes u)\|}{\|Y(y_j \otimes u)\|} \geqslant 1, \qquad \forall u \in \mathbf{C}^N$$

while

$$\frac{\|X(y_j \otimes u_j)\|}{\|Y(y_j \otimes u_j)\|} = 1.$$

Equivalently,

$$\frac{\|XY^{-1}v\|}{\|v\|} \geqslant 1,$$

and

$$\frac{\|XY^{-1}v_j\|}{\|v_j\|} = 1,$$

where $v := Y(y_j \otimes u)$, $v_j := Y(y_j \otimes u_j)$. It follows at once that

$$(XY^{-1})^*(XY^{-1})\, v_j = v_j$$

or equivalently,

$$(Y^{-1})^* X^* X(y_j \otimes u_j) = Y(y_j \otimes u_j),$$

whence

$$X^* X(y_j \otimes u_j) = Y^* Y(y_j \otimes u_j).$$

This last relation is equivalent to the required conclusion of the lemma. ∎

THEOREM 3. *Under the hypotheses of Lemma* 3, *we have*

$$y_j \otimes u_j - \frac{(\rho_j^2 - \bar{\lambda}_j B_j)(\rho_j^2 - B_j^* B_j)^{-1}(\rho_j^2 - \lambda_j B_j^*)}{\rho_j^2 - |\lambda_j|^2} P_{\mathcal{M}_j}(y_j \otimes u_j) \perp y_j \otimes \mathbf{C}^N. \tag{10}$$

Proof. We have that

$$P_{y_j \otimes \mathbf{C}^N}\Big((I - P_{\mathcal{M}_j})\, y_j \otimes u_j - \frac{\rho_j^2}{\rho_j^2 - |\lambda_j|^2} (\lambda_j - B_j)^* (\rho_j^2 - B_j B_j^*)^{-1}$$
$$\times (\lambda_j - B_j)\, P_{\mathcal{M}_j}(y_j \otimes u_j) \Big) = 0$$

so we only need show that

$$(\rho_j^2 - |\lambda_j|^2) + \rho_j^2(\bar{\lambda}_j - B_j^*)(\rho_j^2 - B_j B_j^*)^{-1}(\lambda_j - B_j)$$
$$= (\rho_j^2 - \bar{\lambda}_j B_j)(\rho_j^2 - B_j^* B_j)^{-1}(\rho_j^2 - \lambda_j B_j^*).$$

Indeed, using the identity

$$(\rho_j^2 - B_j B_j^*)^{-1} B_j = B_j(\rho_j^2 - B_j^* B_j)^{-1},$$

and setting

$$\beta := -\rho_j^2 \bar{\lambda}_j(\rho_j^2 - B_j B_j^*)^{-1} B_j - \rho_j^2 \lambda_j B_j^*(\rho_j^2 - B_j B_j^*)^{-1}$$
$$= -\rho_j^2 \bar{\lambda}_j B_j(\rho_j^2 - B_j^* B_j)^{-1} - \rho_j^2 \lambda_j(\rho_j^2 - B_j^* B_j)^{-1} B_j^*,$$

we obtain that

$$(\rho_j^2 - |\lambda_j|^2) - \rho_j^2(\bar{\lambda}_j - B_j^*)(\rho_j^2 - B_j B_j^*)^{-1}(\lambda_j - B_j)$$
$$= \rho_j^2 |\lambda_j|^2(\rho_j^2 - B_j B_j^*)^{-1} + \rho_j^2 B_j^*(\rho_j^2 - B_j B_j^*)^{-1} B_j + \rho_j^2 - |\lambda_j|^2 + \beta$$
$$= (\rho_j^2(\rho_j^2 - B_j B_j^*)^{-1} - 1)|\lambda_j|^2 + (B_j^*(\rho_j^2 - B_j B_j^*)^{-1} B_j + 1)\,\rho_j^2 + \beta$$
$$= B_j B_j^* |\lambda_j|^2(\rho_j^2 - B_j B_j^*)^{-1} + ((\rho_j^2 - B_j^* B_j)^{-1} B_j^* B_j + 1)\,\rho_j^2 + \beta$$
$$= B_j |\lambda_j|^2(\rho_j^2 - B_j^* B_j)^{-1} B_j^* + \rho_j^4(\rho_j^2 - B_j^* B_j)^{-1} + \beta$$
$$= (\rho_j^2 - \bar{\lambda}_j B_j)(\rho_j^2 - B_j^* B_j)^{-1}(\rho_j^2 - \lambda_j B_j^*).$$

This completes the proof of the theorem. ∎

Now we can write out a rather explicit expression for (10) using linear algebra. To do this, let us denote by Γ_j the self-adjoint (Grammian) matrix given by

$$(\Gamma_j)_{ik} = \langle y_k \otimes u_k, y_i \otimes u_i \rangle, \qquad 1 \leqslant i, k \leqslant n, i, k \neq j.$$

Let Λ_j be the $(n-1) \times (n-1)$ diagonal matrix defined by

$$(\Lambda_j)_{ik} = \lambda_i \, \delta_{ik}, \qquad 1 \leqslant i, k \leqslant n,\ i, k \neq j,$$

where δ_{ik} denotes the Kronecker delta. Note that Λ_j is precisely the matrix of B_j in the basis $\{y_i \otimes u_i : 1 \leqslant i \leqslant n,\ i \neq j\}$. Moreover, it is a standard fact in linear algebra that the matrix of B_j^* in this basis is $\Gamma_j^{-1} \Lambda_j^* \Gamma_j$.

Write

$$P_{\mathcal{M}_j}(y_j \otimes u_j) = \sum_{i \neq j} \alpha_i^{(j)} y_i \otimes u_i.$$

To calculate $\alpha_i^{(j)}$, we note that

$$\langle y_j \otimes u_j - P_{\mathcal{M}_j}(y_j \otimes u_j),\ y_k \otimes u_k \rangle = 0$$

for $k \neq j$, so that

$$\mu_k^{(j)} := \frac{\langle u_j, u_k \rangle}{1 - \bar{z}_k z_j} = \sum_{i \neq j} \alpha_i^{(j)} (\Gamma_j)_{ki}$$

and hence we have that

$$\alpha^{(j)} = \Gamma_j^{-1} \mu^{(j)},$$

where $\alpha^{(j)}$ denotes the column vector with components $\alpha_i^{(j)}$, and similarly for $\mu^{(j)}$.

Next set

$$(\rho_j^2 - \bar{\lambda}_j B_j)(\rho_j^2 - B_j^* B_j)^{-1} (\rho_j^2 - \lambda_j B_j^*)\, P_{\mathcal{M}_j}(y_j \otimes u_j) = \sum_{i \neq j} \eta_i^{(j)} y_i \otimes u_i.$$

Using the preceding computations, the column vector $\eta^{(j)}$ with components $\eta_i^{(j)}$ can be explicitly calculated as follows:

$$\begin{aligned}\eta^{(j)} &= (\rho_j^2 - \bar{\lambda}_j A_j)(\rho_j^2 - \Gamma_j^{-1} A_j^* \Gamma_j A_j)^{-1} (\rho_j^2 - \lambda_j \Gamma_j^{-1} A_j^* \Gamma_j)\, \Gamma_j^{-1} \mu^{(j)} \\ &= (\rho_j^2 - \bar{\lambda}_j A_j)(\rho_j^2 - \Gamma_j^{-1} A_j^* \Gamma_j A_j)^{-1} \Gamma_j^{-1} (\rho_j^2 - \lambda_j A_j^*)\, \mu^{(j)} \\ &= (\rho_j^2 - \bar{\lambda}_j A_j)(\rho_j^2 \Gamma_j - A_j^* \Gamma_j A_j)^{-1} (\rho_j^2 - \lambda_j A_j^*)\, \mu^{(j)}. \qquad (11)\end{aligned}$$

We can now write out the conclusion of Theorem 3 in a rather simple form. Indeed we have shown that

$$(\rho_j^2 - |\lambda_j|^2)\, y_j \otimes u_j + \sum_{i \neq j} \eta_i^{(j)} y_i \otimes u_i \perp y_j \otimes \mathbf{C}^N,$$

where the $\eta_i^{(j)}$ are computed from Eq. (11). But from this it is easy to compute that in fact

$$\sum_{i \neq j} \frac{\eta_i^{(j)} u_i}{1 - \bar{z}_j z_i} + \frac{\rho_j^2 - |\lambda_j|^2}{1 - |z_j|^2} u_j = 0. \qquad (12)$$

The above argument then proves that if $\rho_j > \max\{\|B_j\|, |\lambda_j|\}$, then ρ_j and u_j satisfy the system (12). This is precisely Theorem 3 stated in matrix form.

It is possible to rewrite Eqs. (12) in a slightly modified form and to thereby extend their validity to the case in which

$$\rho_j = \|B_j\| > |\lambda_j|.$$

Set

$$\begin{aligned}\mathcal{D}_j &:= \rho_j^2 - \lambda_j A_j^*, \\ \mathcal{N}_j &:= \rho_j^2 \Gamma_j - A_j^* \Gamma_j A_j.\end{aligned}$$

Then if we multiply both sides of Eq. (12) by $\det \mathcal{N}_j$, we obtain that

$$\det \mathcal{N}_j \sum_{i \neq j} \frac{\eta_i^{(j)} u_i}{1 - \bar{z}_j z_i} + \det \mathcal{N}_j \frac{\rho_j^2 - |\lambda_j|^2}{1 - |z_j|^2} u_j = 0. \tag{13}$$

But

$$\det \mathcal{N}_j \cdot \mathcal{D}_j^* \mathcal{N}_j^{-1} \mathcal{D}_j \mu^{(j)} = \mathcal{D}_j^* \mathcal{N}_j^{\mathrm{alg}} \mathcal{D}_j u^{(j)},$$

where $\mathcal{N}_j^{\mathrm{alg}}$ denotes the algebraic adjoint of $\mathcal{N}_j$, that is, $\mathcal{N}_j^{\mathrm{alg}} \mathcal{N}_j = \det \mathcal{N}_j) I$. Therefore

$$\sum_{i \neq j} \frac{(\mathcal{D}_j^* \mathcal{N}_j^{\mathrm{alg}} \mathcal{D}_j \mu^{(j)})_i u_i}{1 - \bar{z}_j z_i} + (\rho_j^2 - |\lambda_j|^2) \det \mathcal{N}_j u_j = 0.$$

Now

$$\sum_{i \neq j} \frac{(\mathcal{D}_j^* \mathcal{N}_j^{\mathrm{alg}} \mathcal{D}_j \mu^{(j)})_i u_i}{1 - \bar{z}_j z_i}$$
$$= \sum_{i,k \neq j} \frac{(\rho_j^2 - \bar{\lambda}_j \lambda_i)}{1 - \bar{z}_j z_i} (\mathcal{N}_j^{\mathrm{alg}})_{ik} (\rho_j^2 - \lambda_j \bar{\lambda}_k) \frac{\langle u_j, u_k \rangle}{1 - \bar{z}_j z_k} u_i.$$

Thus we see that

$$\sum_{i,k \neq j} \frac{(\rho_j^2 - \bar{\lambda}_j \lambda_i)}{1 - \bar{z}_j z_i} (\mathcal{N}_j^{\mathrm{alg}})_{ik} (\rho_j^2 - \lambda_j \bar{\lambda}_k) \frac{\langle u_j, u_k \rangle}{1 - \bar{z}_j z_k} u_i + \frac{\rho_j^2 - |\lambda_j|^2}{1 - |z_j|^2} u_j = 0. \tag{14}$$

Define now

$$Q_j := Q_j(\rho_j; z_1, ..., z_n; \lambda_1, ..., \lambda_n: u_1, ..., u_n)$$

$$:= \begin{bmatrix} \cdots & \frac{\rho_j^2 - \bar{\lambda}_{j-1} \lambda_1}{1 - \bar{z}_{j-1} z_1} \langle u_1, u_{j-1} \rangle & \frac{\rho_j^2 - \bar{\lambda}_j \lambda_1}{1 - \bar{z}_j z_1} u_1 & \frac{\rho_j^2 - \bar{\lambda}_{j+1} \lambda_1}{1 - \bar{z}_{j+1} z_1} \langle u_1, u_{j+1} \rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \frac{\rho_j^2 - \bar{\lambda}_{j-1} \lambda_n}{1 - \bar{z}_{j-1} z_n} \langle u_n, u_{j-1} \rangle & \frac{\rho_j^2 - \bar{\lambda}_j \lambda_n}{1 - \bar{z}_j z_n} u_n & \frac{\rho_j^2 - \bar{\lambda}_{j+1} \lambda_n}{1 - \bar{z}_{j+1} z_n} \langle u_n, u_{j+1} \rangle & \cdots \end{bmatrix}.$$

Note that Q_j can be regarded as an antilinear operator from $\mathbf{C}^N$ to $B(\mathbf{C}^N)$, given by

$$Q_j v :=$$

$$\begin{bmatrix} \cdots & \frac{\rho_j^2 - \bar{\lambda}_{j-1} \lambda_1}{1 - \bar{z}_{j-1} z_1} \langle u_1, u_{j-1} \rangle & \frac{\rho_j^2 - \bar{\lambda}_j \lambda_1}{1 - \bar{z}_j z_1} \langle u_1, v \rangle & \frac{\rho_j^2 - \bar{\lambda}_{j+1} \lambda_1}{1 - \bar{z}_{j+1} z_1} \langle u_1, u_{j+1} \rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & \frac{\rho_j^2 - \bar{\lambda}_{j-1} \lambda_n}{1 - \bar{z}_{j-1} z_n} \langle u_n, u_{j-1} \rangle & \frac{\rho_j^2 - \bar{\lambda}_j \lambda_n}{1 - \bar{z}_j z_n} \langle u_n, v \rangle & \frac{\rho_j^2 - \bar{\lambda}_{j+1} \lambda_n}{1 - \bar{z}_{j+1} z_n} \langle u_n, u_{j+1} \rangle & \cdots \end{bmatrix}$$

for $v \in \mathbf{C}^N$. Then it is easy to show that the system (14) is equivalent to the system

$$\det Q_j(\rho_j; z_1, ..., z_n; \lambda_1, ..., \lambda_n; u_1, ..., u_n) := 0. \tag{15}$$

Note that Eq. (15) makes sense when $\det \mathcal{N}_j = 0$, which occurs in case $\rho_j = \|B_j\| > |\lambda_j|$. From the above discussion, we can infer:

COROLLARY 1. *If*

$$\rho_{T,\mathcal{M}} > \max_{1 \leqslant j \leqslant n} \{|\lambda_j|\}, \tag{16}$$

then $\rho_{T,\mathcal{M}}$ *and the corresponding optimal* $u_1, ..., u_n$ *satisfy the system*

$$\det Q_j(\rho_{T,\mathcal{M}}; z_1, ..., z_n; u_1, ..., u_n) = 0 \qquad \textit{for} \quad 1 \leqslant j \leqslant n. \tag{17}$$

Proof. We have only explicitly worked out the details of the proof in case

$$\rho_{T,\mathcal{M}} > \max_{1 \leqslant j \leqslant n} \{\|B_j\|, |\lambda_j|\}.$$

A careful perusal of the above computations reveals however that they do indeed extend to the situation when $\rho_{T,\mathcal{M}}$ satisfies the inequality (16). The details are left to the interested reader. ∎

Remark 5. Let

$$\bar{\rho} := \inf\{\|f\|_\infty : f \in H^\infty, f(z_j) = \lambda_j \text{ for } 1 \leqslant j \leqslant n\}.$$

Clearly, we have that

$$\bar{\rho} \geqslant \rho_{T,\mathcal{M}} \geqslant \max_{1 \leqslant j \leqslant n} \{|\lambda_j|\}.$$

Thus in an algorithm for the computation of $\rho_{T,\mathcal{M}}$, one need only look in a finite interval. In fact, we plan to organize the above results into such an algorithm in a future engineering-oriented paper, as well as work out some examples of applied interest.

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